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Anisotropic Cosmological Models of Bianchi Types III and V in Lyra's Geometry

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Exact solutions to Einstein's equations are presented in vacuum and in the presence of stiff matter for spatially homogeneous cosmological models of Bianchi types type III and V in the normal gauge for Lyra's geometry. Solutions represent anisotropic cosmological universes which contract from infinite volume at the initial time singularity $T=0$ to zero volume as $T\to\infty$. Some physical properties of the models are also discussed.

1. INTRODUCTION

Experimental studies of the isotropy of cosmic microwave radiation and speculation about the amount of helium formed at early stages of the universe and many other effects have stimulated theoretical interest in anisotropic cosmological models. Bianchi spaces I–IX play important roles in constructing models of spatially homogeneous cosmologies (Ryan and Shepley, 1975). There is a large body of literature concerning specific Bianchi spaces which contain fluids with specific equations of state. The properties of the space-time continuum are not consistent with the geometry of a flat space, but require Riemannian geometry for their description. Lyra (1951) proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears close resemblance to Weyl geometry. The energy-momentum tensor is not conserved in Lyra's geometry. Halford (1970) developed a cosmological theory within the framework of this geometry which results in nonstatic models of perfect fluid cosmologies.

Recently, Singh and Singh (1991) obtained exact solutions of Einstein's field equations for the anisotropic Bianchi type I model in the normal gauge

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for Lyra's geometry. Since Bianchi type I models are a very special subset of spatially homogeneous models, we here consider the more general Bianchi types III and V models in a similar investigation. Field equations are considered in the case of Bianchi type III and V spaces in Lyra's geometry when the displacement vector is time-dependent. Exact solutions of the field equations are obtained in vacuum and for a stiff matter distribution. The physical and geometrical properties of solutions are also discussed.

2. FIELD EQUATIONS

The field equations in normal gauge for Lyra's manifold are (Sen, 1957)

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\tfrac{3}{2}[\phi_\mu\phi_\nu-\tfrac{1}{2}g_{\mu\nu}\phi_\alpha\phi^\alpha]=-KT_{\mu\nu} \tag{1}$$

Here ϕ_μ is the displacement field vector defined as

$$\phi_\mu=(0,0,0,\beta) \tag{2}$$

where $\beta=\beta(t)$ and other symbols have their usual meanings. The energy-momentum tensor for a perfect fluid distribution is given by

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu-pg_{\mu\nu} \tag{3}$$

together with

$$u_\mu u^\mu=1 \tag{4}$$

where p is the pressure, ρ is the energy density, and u^μ is the 4-velocity vector.

2.1. Bianchi Type III Solutions

The metric of the Bianchi type III class of models is taken in the form

$$ds^2=dt^2-A^2\,dx^2-B^2\,e^{2x}\,dy^2-C^2\,dz^2 \tag{5}$$

where A, B, and C are functions of cosmic time t. In comoving coordinates $u^\mu=\delta^\mu_4$, the field equations to be solved are

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=-Kp-\frac{3}{4}\beta^2 \tag{6}$$

$$\frac{\ddot{C}}{C}+\frac{\ddot{A}}{A}+\frac{\dot{C}\dot{A}}{CA}=-Kp-\frac{3}{4}\beta^2 \tag{7}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{1}{A^2}=-Kp-\frac{3}{4}\beta^2 \tag{8}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}-\frac{1}{A^2}=K\rho+\frac{3}{4}\beta^2 \tag{9}$$

$$\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=0 \tag{10}$$

In addition to these equations, the energy conservation equation gives

$$K\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+\left[K(\rho+p)+\frac{3}{2}\beta^2\right]\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=0 \tag{11}$$

A dot denotes differentiation with respect to t.

A linear combination of equations (6)–(9) gives

$$\frac{\ddot{A}}{A}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{1}{A^2}=\frac{1}{2}K(\rho-p) \tag{12}$$

$$\frac{\ddot{B}}{B}+\frac{\dot{B}}{B}\left(\frac{\dot{C}}{C}+\frac{\dot{A}}{A}\right)-\frac{1}{A^2}=\frac{1}{2}K(\rho-p) \tag{13}$$

$$\frac{\ddot{C}}{C}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)=\frac{1}{2}K(\rho-p) \tag{14}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+2\left(\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right)-\frac{2}{A^2}=\frac{3}{2}K(\rho-p) \tag{15}$$

Equation (10), on integration, leads to $A=bB$. Without loss of generality we take the arbitrary constant b equal to unity.

Using the equation of state $p=(\lambda-1)\rho$, $1\leq\lambda\leq 2$, and defining

$$V^3=A^2C \tag{16}$$

we find that equations (12)–(15) and (11) reduce to

$$\frac{\ddot{A}}{A}-\frac{\dot{A}^2}{A^2}+\frac{3\dot{V}\dot{A}}{VA}-\frac{1}{A^2}=\frac{1}{2}K\rho(2-\lambda)/f \tag{17}$$

$$\frac{\ddot{C}}{C}-\frac{\dot{C}^2}{C^2}+\frac{3\dot{V}\dot{C}}{VC}=\frac{1}{2}K\rho(2-\lambda)/f \tag{18}$$

$$\frac{\ddot{V}}{V}+\frac{2\dot{V}^2}{V^2}-\frac{2}{3A^2}=\frac{3}{2}K\rho(2-\lambda)/f \tag{19}$$

$$K\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+3\left(K\lambda\rho+\frac{3}{2}\beta^2\right)\frac{\dot{V}}{V}=0 \tag{20}$$

Making the scale transformation

$$dt = A^2 C\, dT \tag{21}$$

into equations (17)–(20), we get

$$\frac{1}{V^6}\left(\frac{A''}{A} - \frac{A'^2}{A^2} - A^2C^2\right) = \frac{1}{2} K\rho(2-\lambda)/f \tag{22}$$

$$\frac{1}{V^6}\left(\frac{C''}{C} - \frac{C'^2}{C^2}\right) = \frac{1}{2} K\rho(2-\lambda)/f \tag{23}$$

$$\frac{1}{V^6}\left(\frac{V''}{V} - \frac{V'^2}{V^2} - \frac{2}{3} A^2C^2\right) = \frac{3}{2} K\rho(2-\lambda)/f \tag{24}$$

$$K\rho' + \frac{3}{2}\beta\beta' + 3\left(K\lambda\rho + \frac{3}{2}\beta^2\right)\frac{V'}{V} = 0 \tag{25}$$

where a prime denotes differentiation with respect to T.

It is difficult to obtain general solutions of (22)–(25), so we consider some cases of physical interest.

Case 2.1(a). In empty space ($\rho = p = 0$), equations (22)–(25) reduce to

$$\frac{A''}{A} - \frac{A'^2}{A^2} = A^2C^2 \tag{26}$$

$$\frac{C''}{C} - \frac{C'^2}{C^2} = 0 \tag{27}$$

$$\frac{V''}{V} - \frac{V'^2}{V^2} = \frac{2}{3} A^2C^2 \tag{28}$$

$$\beta' + 3\beta\frac{V'}{V} = 0 \tag{29}$$

Equation (27) yields the solution

$$C = e^{mT} \tag{30}$$

m is an integration constant. Here the unnecessary constants will be eliminated by absorption into the suitably defined coordinates.

Inserting (30) into (26), we obtain

$$(\log A)'' = A^2\, e^{2mT} \tag{31}$$

The general solution of (31) is

$$A = a\, e^{-mT} \operatorname{cosech}(aT) \tag{32}$$

where a is an integration constant. Again, inserting (30) and (32) into (28) and integrating, we find that

$$V^3 = a^2\, e^{-mT} \operatorname{cosech}^2(aT) \tag{33}$$

Equation (29) leads to

$$\beta = \frac{n}{V^3} \tag{34}$$

where n is an integration constant. The constants m, n, and a are related by

$$4(a^2 - m^2) = 3n^2 \tag{35}$$

Case 2.1(b). For a stiff-matter universe ($\lambda = 2$) the solutions of equations (17)–(19) are the same as the empty-space solutions, but equation (20) leads to

$$p = \rho = \frac{d}{4KV^6} - \frac{3}{4K}\beta^2 \tag{36}$$

where d is also an integration constant satisfying

$$4(a^2 - m^2) = d \tag{37}$$

We now discuss some physical features of the solutions. For the geometrical properties and singularity we refer to Hawking and Ellis (1973). Both the vacuum and stiff-matter solutions have the same geometrical properties except for the properties of the pressure and matter density near the singularities. The expansion θ $(= u^{\mu}_{;\mu})$ has the expression

$$\theta = -[m + 2a \coth(aT)] \tag{38}$$

which is infinite at $T = 0$ and monotonically decreasing for $T > 0$. In fact, $\theta \to 0$ as $T \to \infty$. The spatial volume V^3 is also infinite at $T = 0$ and tends to zero as $T \to \infty$. On the other hand, the shear scalar has the value

$$\sigma = \frac{1}{3\sqrt{2}} [17m^2 + 6a^2 \coth(aT) + 20am \coth(aT)]^{1/2} \tag{39}$$

The components of the acceleration vector and the rotation tensor are zero. The fluid is therefore moving irrotationally with shear.

The gauge function β is zero at $T = 0$ and tends to infinity as $T \to \infty$. The pressure and energy density are zero at $T = 0$ and become infinite as $T \to \infty$. Thus, the solution corresponds to a contracting model of the universe. The solution in vacuum has similar properties.

2.2. Bianchi Type V Solutions

The metric for the spatially homogeneous Bianchi type V space-time is

$$ds^2 = dt^2 - X^2\,dx^2 - e^{2x}(Y^2\,dy^2 + Z^2\,dz^2) \tag{40}$$

where X, Y, and Z are cosmic scale functions. The importance of Bianchi type V space-times is due to the fact that the space of constant negative curvature is contained in this class as a special case.

Considering the gauge function β as a function of t only, the field equations (1)–(3) for the line element (40) reduce to

$$\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Y}\dot{Z}}{YZ} - \frac{1}{X^2} = -Kp - \frac{3}{4}\beta^2 \tag{41}$$

$$\frac{\ddot{Z}}{Z} + \frac{\ddot{X}}{X} + \frac{\dot{Z}\dot{X}}{ZX} - \frac{1}{X^2} = -Kp - \frac{3}{4}\beta^2 \tag{42}$$

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} - \frac{1}{X^2} = -Kp - \frac{3}{4}\beta^2 \tag{43}$$

$$\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}\dot{Z}}{YZ} + \frac{\dot{Z}\dot{X}}{ZX} - \frac{3}{X^2} = K\rho + \frac{3}{4}\beta^2 \tag{44}$$

$$\frac{2\dot{X}}{X} - \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} = 0 \tag{45}$$

The conservation equation gives

$$K\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + \left[K(\rho + p) + \frac{3}{2}\beta^2\right]\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = 0 \tag{46}$$

Equation (45), on integration, leads to

$$X^2 = YZ \tag{47}$$

where the integration constant is taken to be unity without loss of generality.

Making the scale transformation

$$dt = X^3\,dT \tag{48}$$

and using the procedure as in Section 2.1, we see that the field equations (41)–(46) reduce to

$$\frac{1}{X^6}\left(\frac{X''}{X} - \frac{X'^2}{X^2}\right) - \frac{2}{X^2} = \frac{1}{2}K\rho(2-\lambda)/f \tag{49}$$

$$\frac{1}{X^6}\left(\frac{Y''}{Y} - \frac{Y'^2}{Y^2}\right) - \frac{2}{X^2} = \frac{1}{2}K\rho(2-\lambda)/f \tag{50}$$

$$\frac{1}{X^6}\left(\frac{Z''}{Z}-\frac{Z'^2}{Z^2}\right)-\frac{2}{X^2}=\frac{1}{2}K\rho(2-\lambda)/f \tag{51}$$

$$K\rho'+\frac{3}{2}\beta\beta'+3\left(K\rho\lambda+\frac{3}{2}\beta^2\right)\frac{X'}{X}=0 \tag{52}$$

where a prime denotes differentiation with respect to T.

Case 2.2(a). In empty space, equations (49)–(52) give

$$\frac{X''}{X}-\frac{X'^2}{X^2}-2X^4=0 \tag{53}$$

$$\frac{Y''}{Y}-\frac{Y'^2}{Y}-2X^4=0 \tag{54}$$

$$\frac{Z''}{Z}-\frac{Z'^2}{Z^2}-2X^4=0 \tag{55}$$

$$\beta'+3\beta\frac{X'}{X}=0 \tag{56}$$

Combining (54) and (55), we obtain

$$\frac{Y''}{Y}-\frac{Y'^2}{Y^2}=\frac{Z''}{Z}-\frac{Z'^2}{Z^2} \tag{57}$$

which has the general solution

$$Y=Z\,e^{lT} \tag{58}$$

where l is an integration constant. In view of (58), the solutions of (53)–(56) are

$$X^2=\frac{k}{2}\operatorname{cosech}(kT) \tag{59}$$

$$Y^2=l_1\,e^{lT}\operatorname{cosech}(kT) \tag{60}$$

$$Z^2=l_2\,e^{-lT}\operatorname{cosech}(kT) \tag{61}$$

$$\beta=\frac{l_3}{X^3} \tag{62}$$

where k, l_1, l_2, and l_3 are integration constants satisfying the constraints

$$k^2=4l_1l_2, \qquad 3k^2-l^2=3l_3^2 \tag{63}$$

Case 2.2(b). For stiff-matter-filled space ($p=\rho$), equations (49)–(51) have the same solutions as in empty space, but equation (52) has the solution

$$\rho=\frac{l_4}{4KX^6}-\frac{3}{4K}\beta^2 \tag{64}$$

together with the constraint

$$3k^2-l^2=l_4 \tag{65}$$

The physical behaviors of the solutions are the same as those of Bianchi type III solutions.

3. CONCLUSIONS

Here we have presented exact solutions to Einstein's equations in vacuum and in the presence of stiff matter for spatially homogeneous cosmological models of Bianchi types III and V in normal gauge for Lyra's geometry. These solutions represent anisotropic contracting universes from infinite volume at the initial singularity $T=0$ to zero volume as $T\to\infty$.

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